Dedekind Sums with Equal Values

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Overview

The Dedekind sum $s(a,b)$ for $(a,b)=1$ is defined by,

$$s(a,b)=\sum_{i=1}^{b-1}\left(\left(\frac{i}{b}\right)\right)\left(\left(\frac{ai}{b}\right)\right)$$

where $((\,))$ denotes the sawtooth function,

$$((x))=\begin{cases} x-\lfloor x\rfloor-\frac{1}{2} & \text{if } x\notin\mathbb{Z} \\ 0 & \text{if } x\in\mathbb{Z} \end{cases}$$

We would like to be able to characterize a_1, a_2 such that $s(a_1,b)=s(a_2,b)$

Overview

1. Conjectures for when b is a prime power in $s(a, b)$
2. Generating Functions I
3. Generating Functions II
4. On the Number of Inversions

Preliminary Results and Facts

1. $s(a,b) = s(a+kb,b)$ for $k \in \mathbb{Z}$.

$$s(a+kb,b) = \sum_{i=1}^{b-1} \left(\left(\frac{i}{b}\right)\right)\left(\left(\frac{(a+kb)i}{b}\right)\right) = \sum_{i=1}^{b-1} \left(\left(\frac{i}{b}\right)\right)\left(\left(\frac{ai}{b} + \frac{kbi}{b}\right)\right)$$
$$= \sum_{i=1}^{b-1} \left(\left(\frac{i}{b}\right)\right)\left(\left(\frac{ai}{b}\right)\right) = s(a,b)$$

2. $s(a_1,b) = s(a_2,b)$ if $a_1 a_2 \equiv 1 \pmod{b}$.

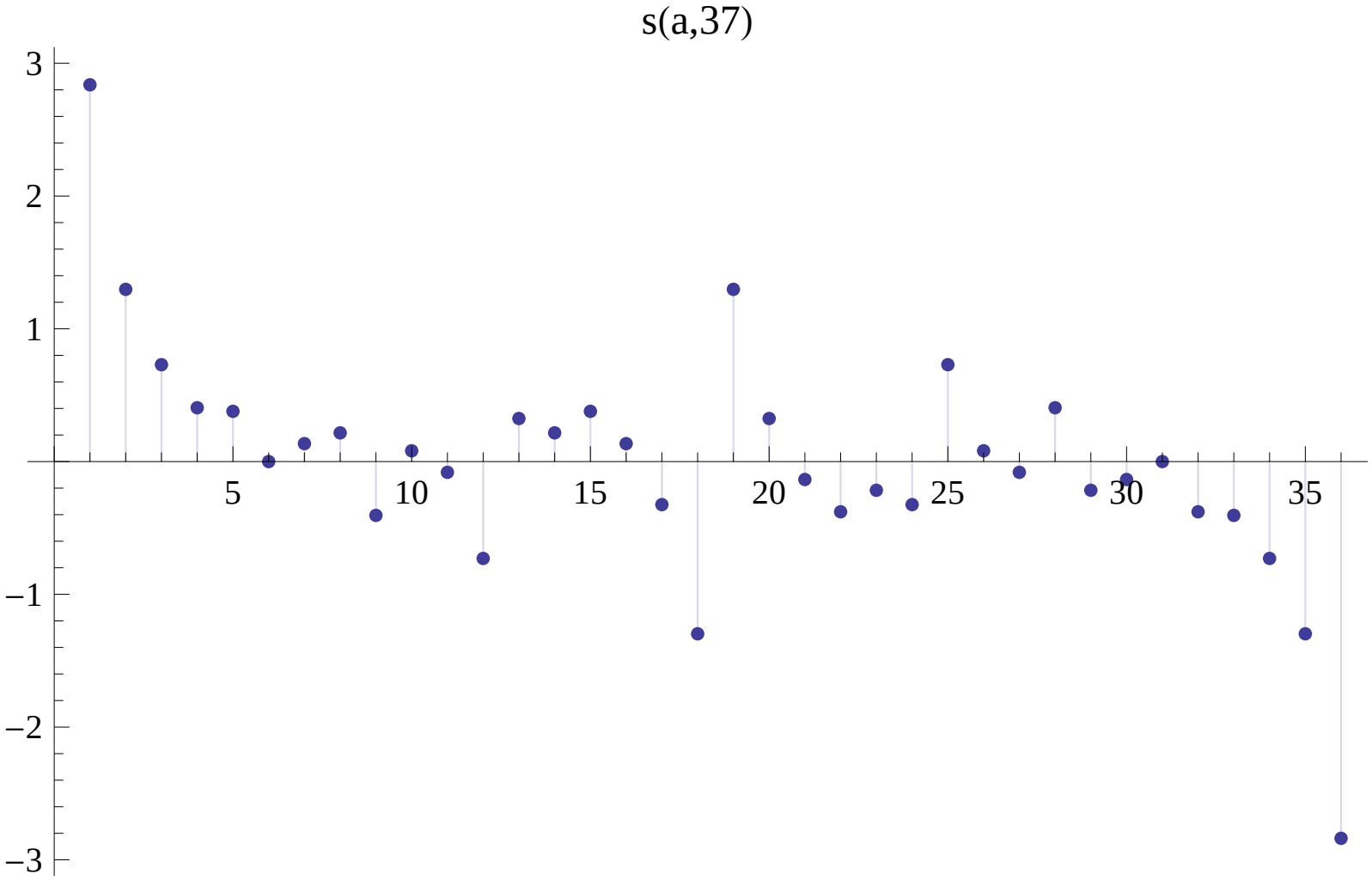
$$s(a_1,b) = \sum_{i=1}^{b-1} \left(\left(\frac{i}{b}\right)\right)\left(\left(\frac{a_1 i}{b}\right)\right) = \sum_{i=1}^{b-1} \left(\left(\frac{a_2 i}{b}\right)\right)\left(\left(\frac{a_2 a_1 i}{b}\right)\right) = s(a_2,b)$$

3. $$s(a,b) + s(b,a) = -\frac{1}{4} + \frac{1}{12}\left(\frac{a}{b} + \frac{1}{ab} + \frac{b}{a}\right)$$

Remark: This allows us to compute Dedekind Sums fast.

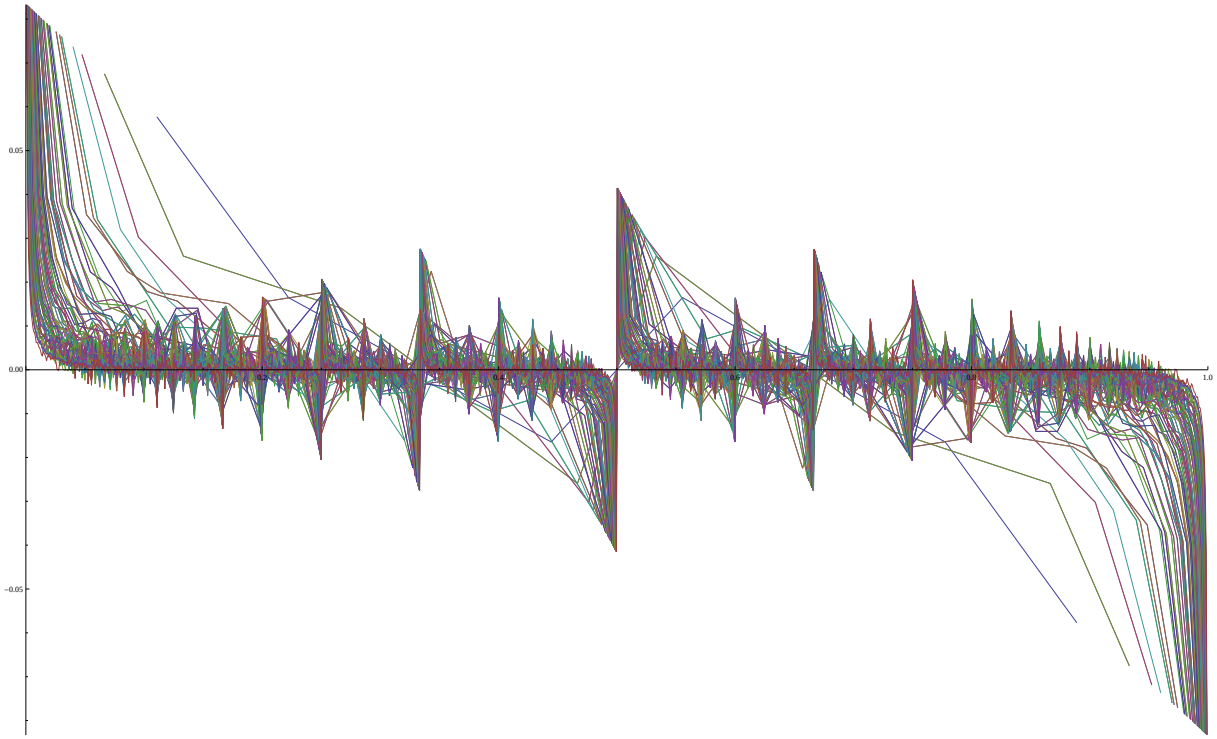
Visualization Strategies

Here's a simple graph of what $s(a, 37)$ looks like,



Visualization Strategies

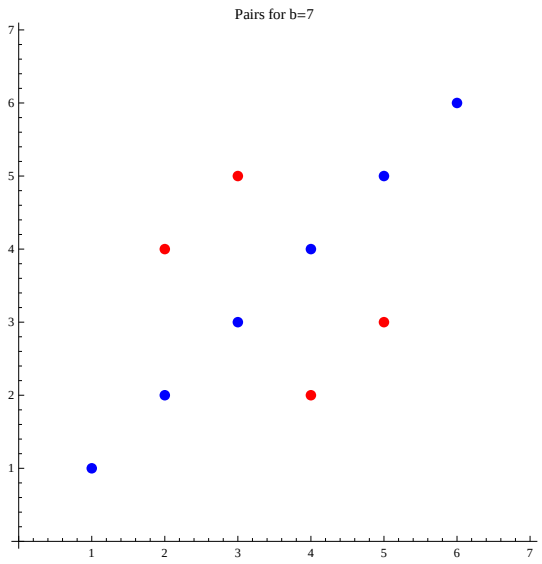
Here's a graph of many values of b (divided by b on both axes),

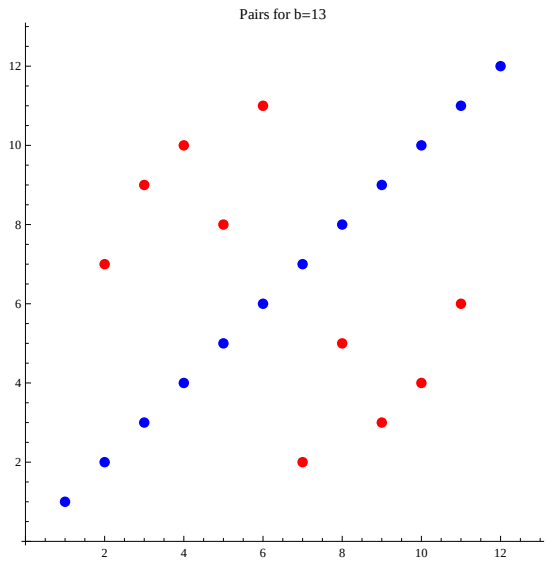


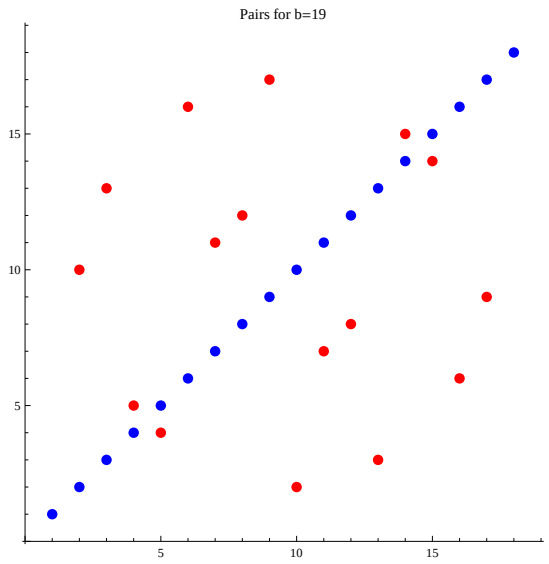
This gives a few insights especially relating to the highest/n–th highest values of the function. But doesn't tell us much about the equality.

Visualization Strategies (b is prime)

But then we came up with another visualization which was somewhat more fruitful for a while,







Each of the axes range up to b.

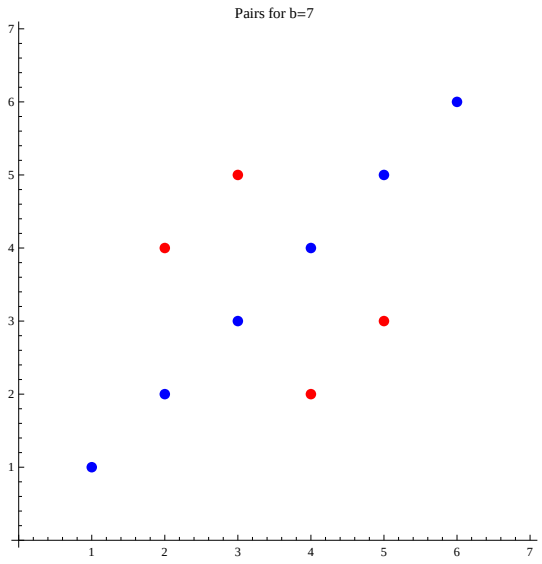
A Theorem by Jabuka, Robins and Wang

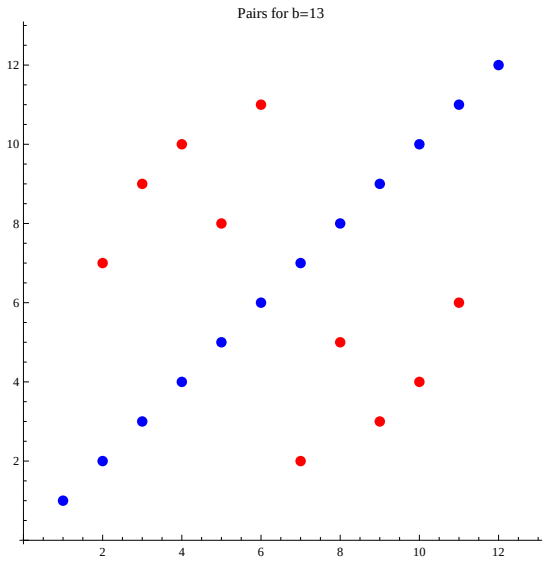
Let b be a positive integer and a_1, a_2 any two integers that are relatively prime to b. If $s(a_1, b) = s(a_2, b)$, then $b \mid (a_1 a_2 - 1)(a_1 - a_2)$.

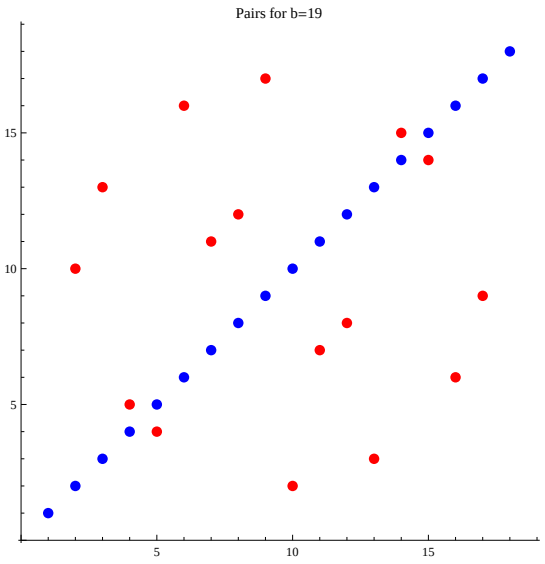
This is gives us the corollary that when b is prime,
$s(a_1, b) = s(a_2.b) \iff a_1 \equiv a_2 \pmod{b}$ or if $a_1 a_2 \equiv 1 \pmod{b}$.

So all equal Dedekind Sums can be explained by the previous two one line proofs when b is prime. Let's take a look at our old picture again.

Prime Case Explained



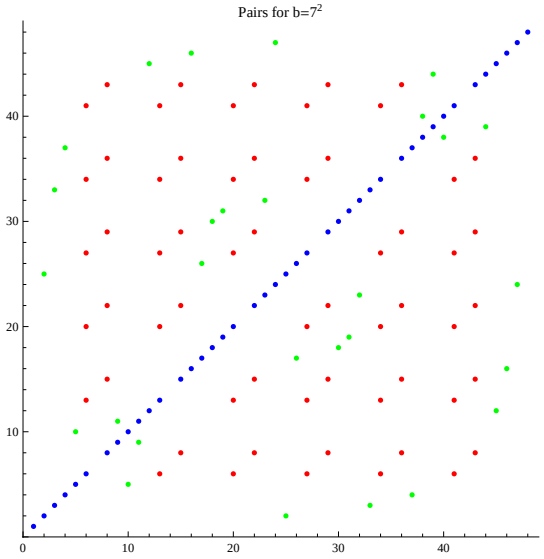


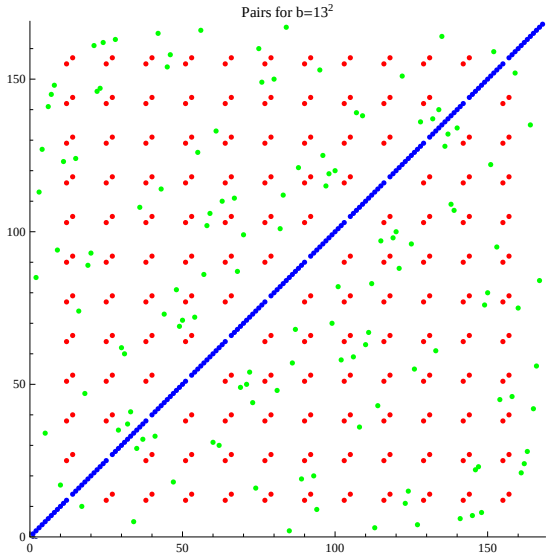


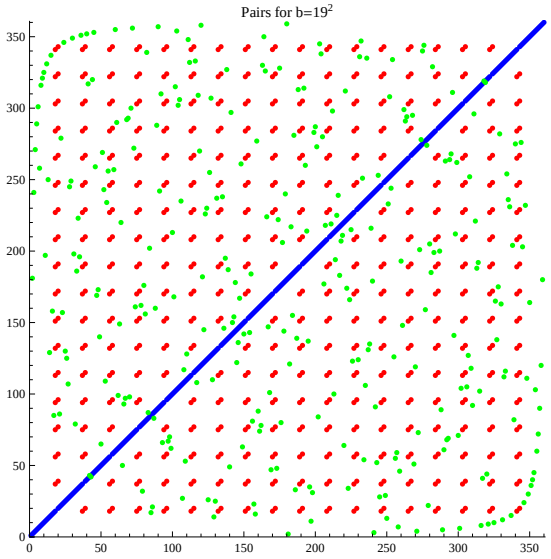
The red dots correspond to pairs which are inverses mod b and the blue ones obviously correspond to ones that are the same mod b.

What About the $b = p^2$?

We looked at the same kinds of graphs again,



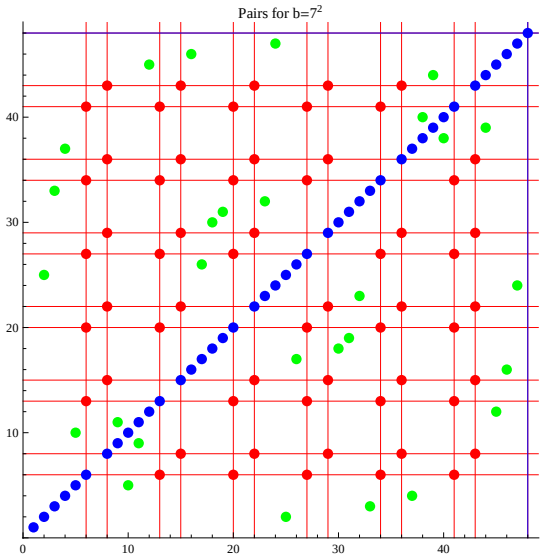


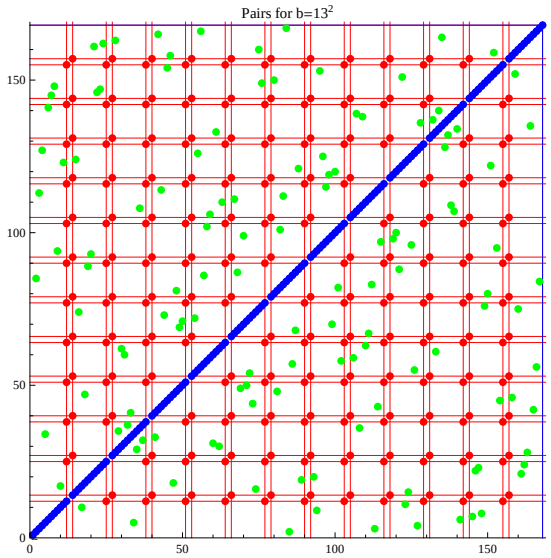


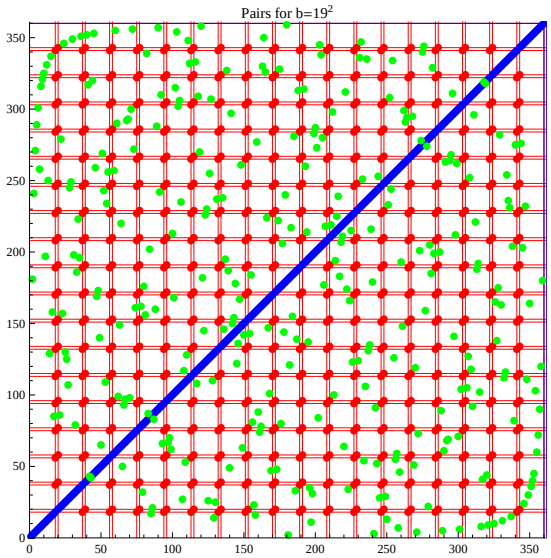
The green dots correspond to pairs which are inverses mod b and the blue ones correspond to ones that are the same mod b. And the red dots are the ones which are aren't in either of those cases.

Red Dots

They look suspiciously well organized, so we were able to guess what lines they lie on.







The lines in each graph are $kp \pm 1$ for $0 < k < p$ and $k \in \mathbb{Z}$.

Coming Up With Formulae

This led us to the conjecture:
$s(a_1, p^2) = s(a_2, p^2)$ if and only if one of the following is true:
(i) $a_1 = a_2 \pmod{p^2}$,
(ii) $a_1 = a_2^{-1} \pmod{p^2}$,
(iii) $a_1 = a_2 = \pm 1 \pmod{p}$.

That is, the non–trivial pairs can be given by,

$$s(kp-1, p^2) = \frac{(p-1)(p+1)}{6p^2} = \frac{1}{6} - \frac{1}{6p^2},$$

and

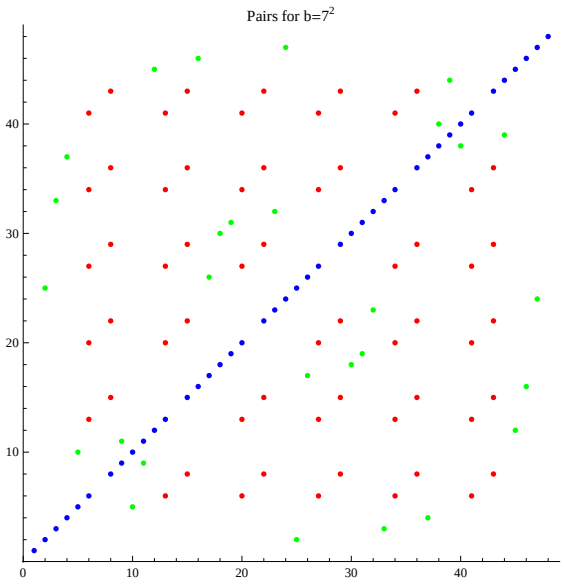
$$s(kp+1, p^2) = -\frac{(p-1)(p+1)}{6p^2} = -\frac{1}{6} + \frac{1}{6p^2}.$$

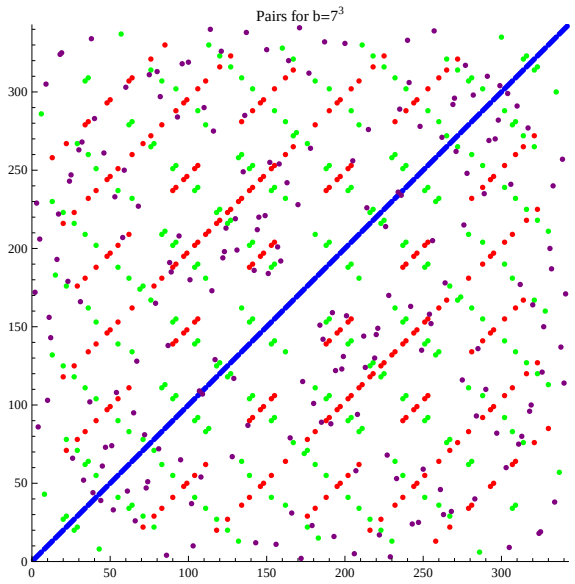
Coming Up With Formulae

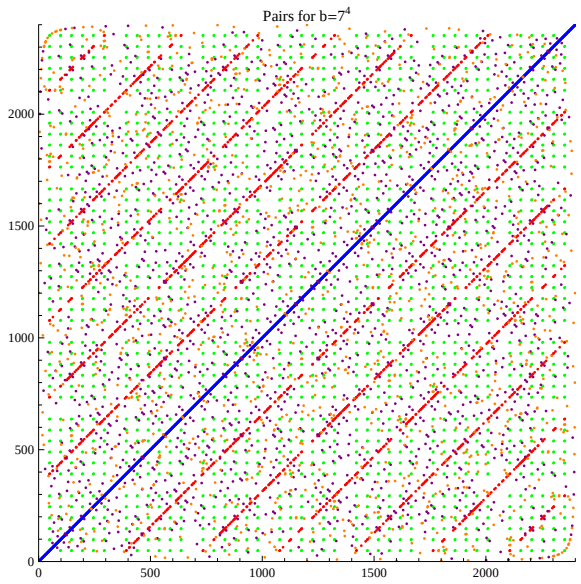
From $p^n|(a_1 - a_2)(a_1 a_2 - 1)$, we know

$$a_1 = a_2 \pmod{p^k} \text{ and } a_1 = a_2^{-1} \pmod{p^{n-k}}$$

for some k.







For $b = p^n$, some nontrivial pairs are given by

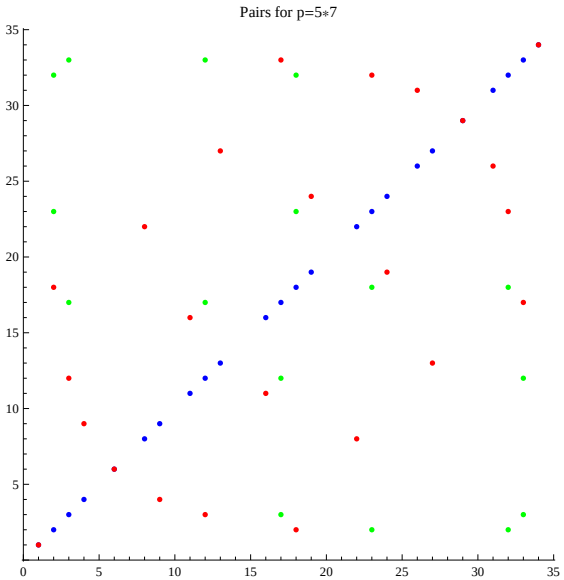
$$s(kp^{n-m} - 1, p^n) = -\frac{p^{2n-2m} - 3p^n + 2}{12p^n} = \frac{1}{4} - \frac{p^{n-2m}}{12} - \frac{1}{6p^n},$$

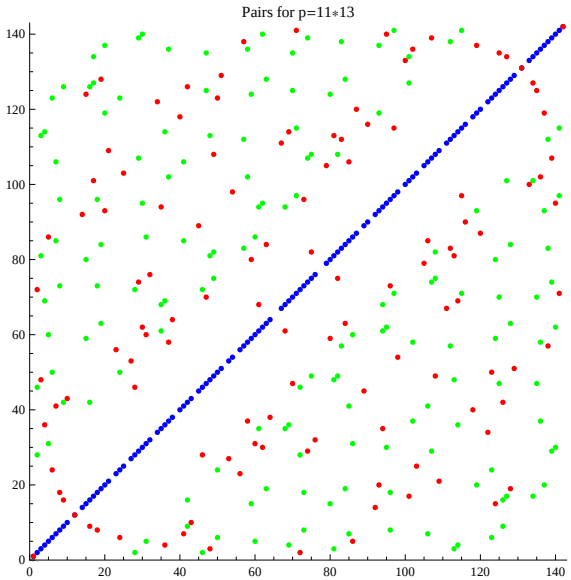
and

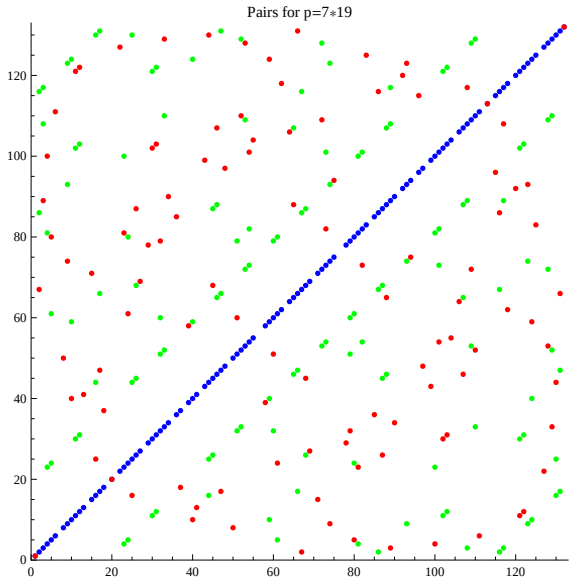
$$s(kp^{n-m} + 1, p^n) = \frac{p^{2n-2m} - 3p^n + 2}{12p^n} = -\frac{1}{4} + \frac{p^{n-2m}}{12} + \frac{1}{6p^n}$$

for $1 \leq m \leq n/2$ and $(k, p) = 1$.

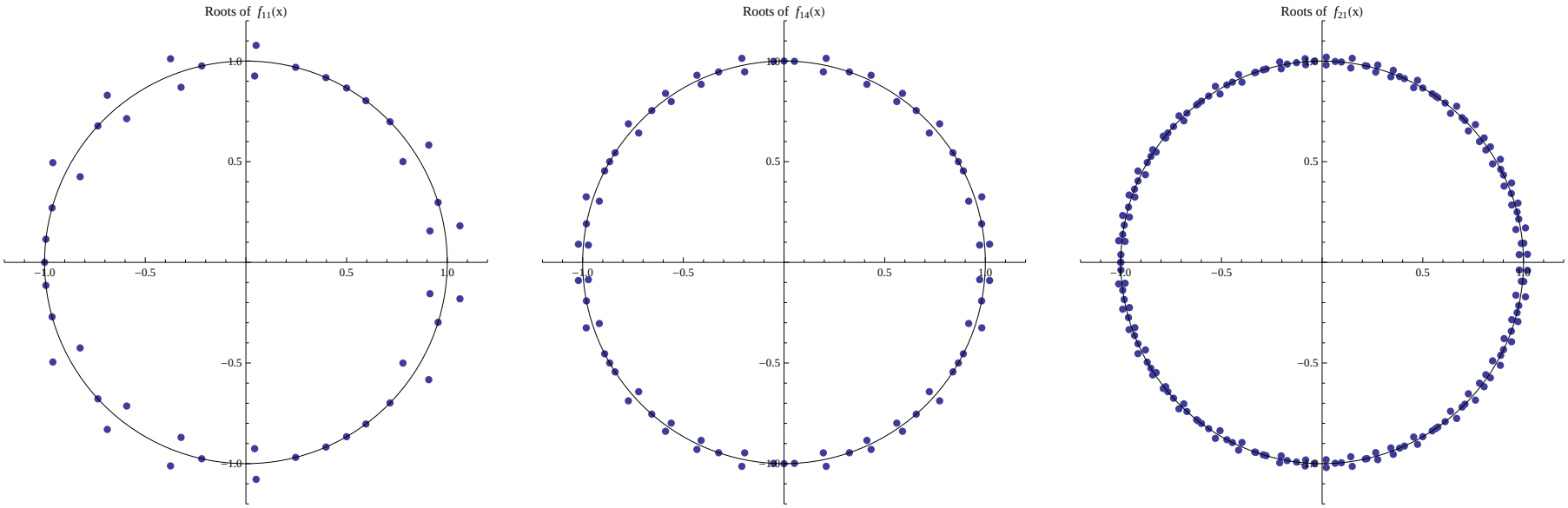
For the $b = pq$ case, there were no clear patterns.





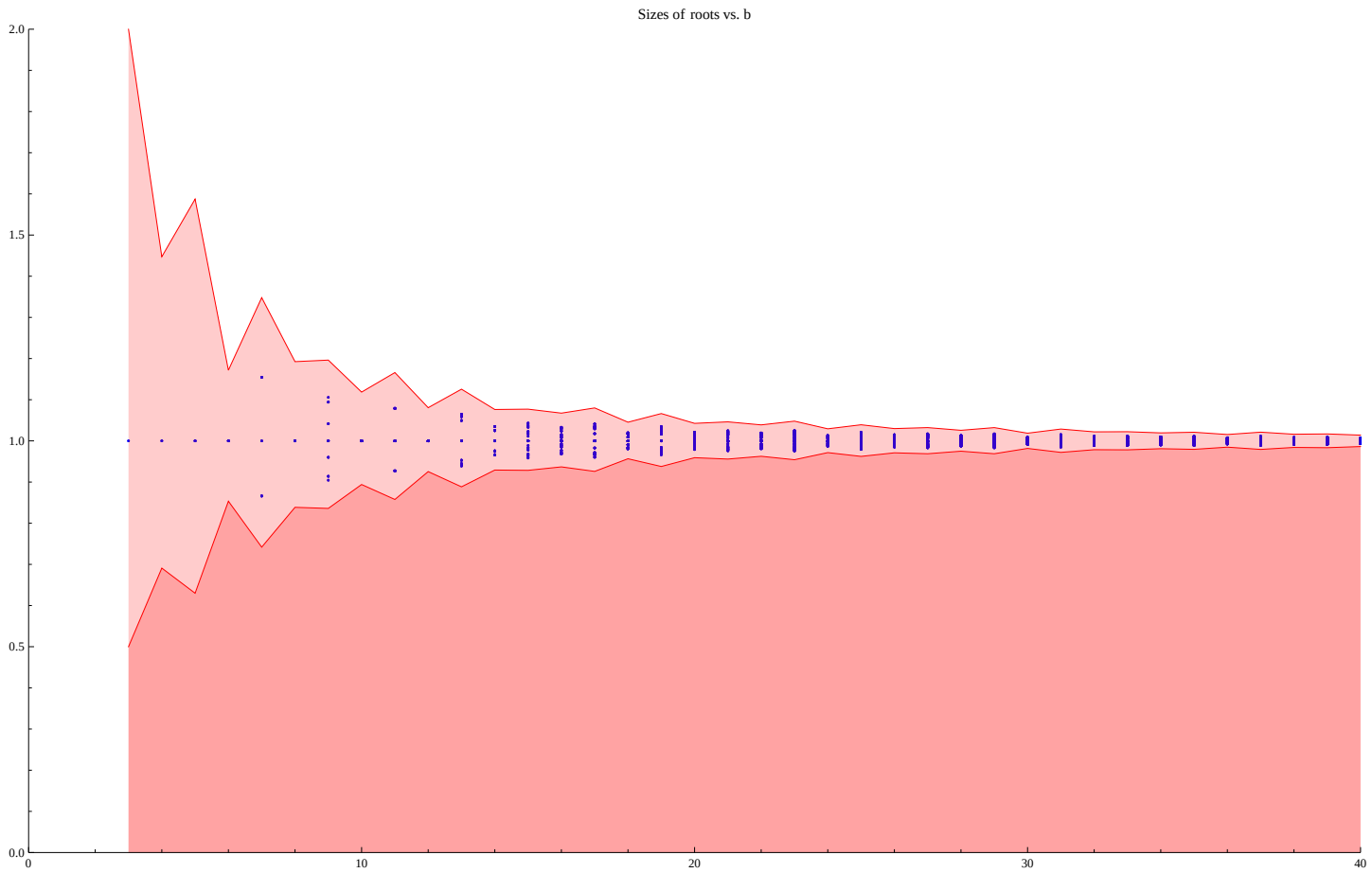


Some roots of the polynomial on the complex plane for various b:



Some are on the unit circle, and some are roots of unity.

A plot of sizes of roots (blue dots) for several values of b:



They are bounded by

$$e^{-\frac{8\log\varphi(b)}{b^2-1}} < x < e^{\frac{8\log\varphi(b)}{b^2-1}}$$

Some factorizations of the polynomial:

b	$f_b(x)$
2	1
3	$1+x$
4	$(1+x)(1-x+x^2)$
5	$(1+x)^2(1-x+x^2)^2$
6	$(1+x^2)(1-x^2+x^4-x^6+x^8)$
7	$(1+x)(1-x+x^2)(1-x^3+3x^6-x^9+x^{12})$
8	$(1+x)(1-x+x^2)(1+x^4)(1-x^3+x^6)(1-x^4+x^8)$
9	$1+2x^{10}+2x^{18}+x^{28}$
10	$(1+x^2)^2(1-x^2+x^4)^2(1-x^6+x^{12})^2$
11	$(1+x)(1-x+x^2)(1-x^3+x^6-x^9+x^{12}+x^{15}+x^{18}-x^{21}+x^{24}+x^{27}+x^{30}-x^{33}+x^{36}-x^{39}+x^{42})$
12	$(1+x)(1-x+x^2)(1+x^4)(1-x^3+x^6)(1-x^9+x^{18})(1-x^4+x^8-x^{12}+x^{16}-x^{20}+x^{24})$
13	$(1+x)^2(1-x+x^2)^2(1-2x^3+3x^6-4x^9+5x^{12}-6x^{15}+7x^{18}-6x^{21}+5x^{24}-4x^{27}+5x^{30}-4x^{33}+5x^{36}$ $-6x^{39}+7x^{42}-6x^{45}+5x^{48}-4x^{51}+3x^{54}-2x^{57}+x^{60})$
14	$(1+x^2)(1-x^2+x^4)(1-x^6+x^{12}-x^{18}+x^{24}+x^{30}-x^{36}+x^{42}+x^{48}-x^{54}+x^{60}-x^{66}+x^{72})$

Proposition. $(x+1)|f_b(x)$ precisely when b is odd and not a square or when $4|b$.

Proof.

If b is odd, then $(-1)^{\mathrm{inv}(a,b)} = \left(\frac{a}{b}\right)$. Hence,

$$f_b(-1) = \sum_{(a,b)=1} (-1)^{\mathrm{inv}(a,b)} = \sum_{(a,b)=1} \left(\frac{a}{b}\right) = \begin{cases} 0 & \text{if } b \text{ is not square} \\ \varphi(b) & \text{if } b \text{ is square} \end{cases}$$

When b is even, it turns out

$$(-1)^{\mathrm{inv}(a,b)} = \begin{cases} (-1)^{\frac{a-1}{2}} & \text{if } b = 0 \pmod 4 \\ 1 & \text{if } b = 2 \pmod 4 \end{cases}$$

from which it follows that

$$f_b(-1) = \begin{cases} 0 & \text{if } b = 0 \pmod 4 \\ \varphi(b) & \text{if } b = 2 \pmod 4 \end{cases}$$

Other facts:

1. If b is odd, not a square, and not divisible by 3, then $(x^3+1)|f_b(x)$.
2. If b is not a square and is $1 \pmod 4$, then $(x+1)^2|f_b(x)$. If b is also not divisible by 3, then $(x^3+1)^2|f_b(x)$.
3. If $b = ck$ is odd, c is not a square, and $(c,k)=1$, then $f_b(\zeta_{2k})=0$. If b is also not divisible by 3, then $f_b(\zeta_{6k})=0$.

Conjecture. For $b = ck$, ζ_{2k} and ζ_{6k} are roots of $f_b(x)$ precisely under the following conditions:
(i) If $c = 2 \pmod 4$, then ζ_{2k} and ζ_{6k} are roots of $f_b(x)$ precisely when $k = 4 \pmod 8$.
(ii) If $c = 0 \pmod 4$, then ζ_{2k} and ζ_{6k} are roots of $f_b(x)$ precisely when $k \neq 2 \pmod 4$ and $8 \nmid k$.
(iii) If $c = 3^m n$, $m \geq 1$, and $(n, 6) = 1$, then only ζ_{2k} is a root of $f_b(x)$ precisely when $3n \nmid k$ and c is not a square.
(iv) If $(c, 6) = 1$, then ζ_{2k} and ζ_{6k} are roots of $f_b(x)$ precisely when $c \nmid k$ and c is not a square.

In trying to understand this conjecture, we discovered that the polynomial takes values of Kloosterman sums:

$$K_m(x,y) = \sum_{(a,m)=1} e^{\frac{2\pi i}{m}(xa+ya^{-1})}$$

If $4k|b$, then

$$f_b\left(e^{\frac{\pi i}{k}}\right) = \frac{1}{2}e^{\frac{\pi i}{2k}}K_{2b}\left(\frac{b}{4k},\frac{b}{4k}\right)$$

If $2k|b$ but $4k \nmid b$, then

$$f_b\left(e^{\frac{\pi i}{k}}\right) = \frac{1}{4}ie^{\frac{\pi i}{2k}}K_{4b}\left(\frac{b}{2k},\frac{b}{2k}(1-b)\right)$$

The conjecture seems to cover all cases when ζ_{2k} or ζ_{6k} is a root for $k|b$. There are rarer cases when $k \nmid b$ which are more mysterious.

b	Unexplained roots	b	Unexplained roots
8	ζ_{18}	117	ζ_{8}
18	ζ_{16}	136	ζ_{18}, ζ_{306}
22	ζ_{20}, ζ_{60}	138	ζ_{108}
26	ζ_{20}, ζ_{60}	148	ζ_{18}, ζ_{36}
29	ζ_{18}	173	ζ_{18}
40	ζ_{18}, ζ_{90}	186	ζ_{20}
45	ζ_{8}, ζ_{40}	198	ζ_{20}
46	ζ_{36}	200	ζ_{18}
56	ζ_{18}	204	ζ_{54}
57	ζ_{54}	296	ζ_{18}
70	ζ_{18}, ζ_{90}	317	ζ_{18}
74	ζ_{36}	325	ζ_{18}, ζ_{90}
80	ζ_{14}, ζ_{42}	332	ζ_{72}
83	ζ_{18}	345	ζ_{54}
114	ζ_{108}	362	ζ_{36}
117	ζ_{8}	398	ζ_{18}

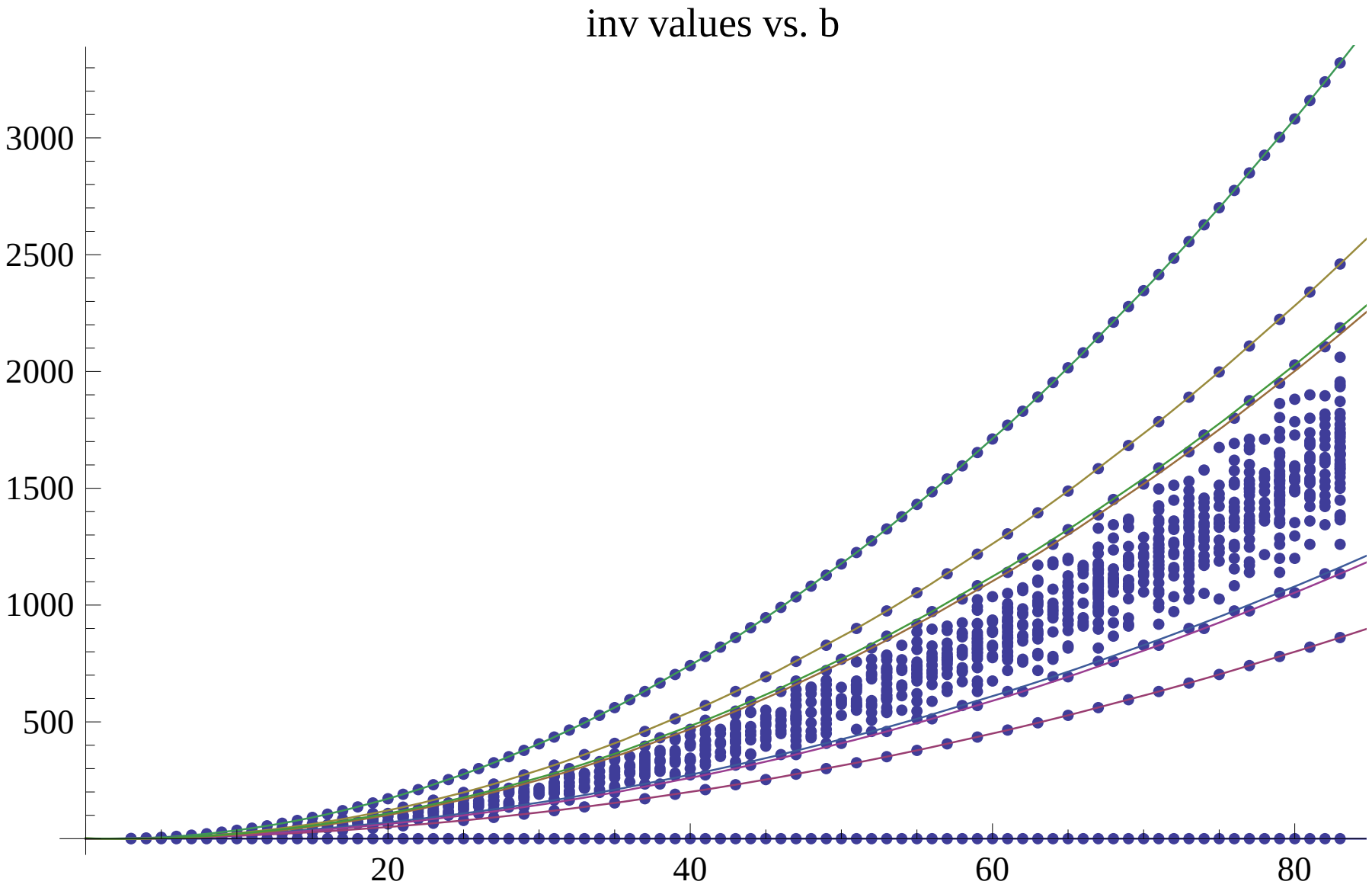
If we have the following sequence

$$r_{-1} = b, r_0 = a, r_1, r_2, r_2, \ldots, r_n, r_{n+1} = 1$$

where r_{j+2} is r_j reduced modulo r_{j+1} for all $-1 \leq j \leq n-1$, then

$$\mathrm{inv}(a,b) = \frac{a-1}{4a}b^2 + \left(\frac{(a-1)(a-2)}{4a} + \frac{1}{4}\sum_{j=1}^{n}\frac{(-1)^j}{r_j r_{j-1}}(r_j - 1)(r_{j-1} - 1)(r_j + r_{j-1} - 1)\right)b - \frac{(a-1)^2}{4a}$$

This means that every inv value lies on a parabola in b.


inv values vs. b
3000
2500
2000
1500
1000
500
20
40
60
80

Proposition. If $2 \leq a \leq b-2$, then

$$\frac{b^2-1}{8} \leq \operatorname{inv}(a,b) \leq \frac{3b^2-12b+9}{8}.$$

Proof. The proof is by induction on b. There are three cases:

1. If $b \neq \pm 1 \pmod{a}$, then use the induction hypothesis:

$$\operatorname{inv}(a,b) \geq -\frac{1}{8}(b+2)a + \frac{1}{4}(b^2+3b+2) - \frac{1}{8}(2b^2+5b+2)\frac{1}{a}.$$

2. If $b = 1 \pmod{a}$, then by reciprocity,

$$\operatorname{inv}(a,b) = \frac{1}{4}(b-1)a + \frac{1}{4}(b^2-3b+2) - \frac{1}{4}(b^2-2b+1)\frac{1}{a}.$$

3. If $b = -1 \pmod{a}$, then again by reciprocity,

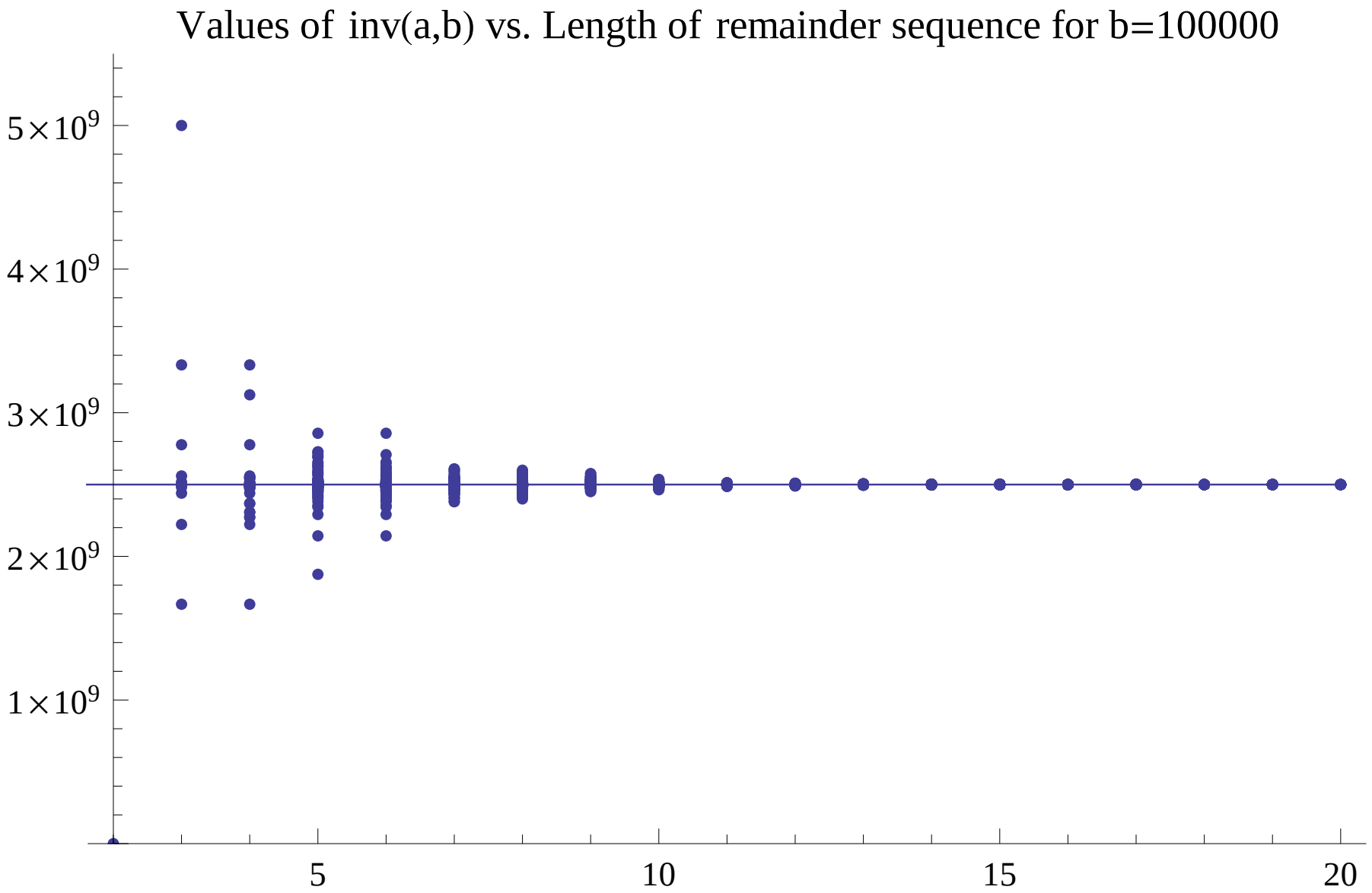
$$\operatorname{inv}(a,b) = -\frac{1}{4}(b+1)a + \frac{1}{4}(b^2+3b+2) - \frac{1}{4}(b^2+2b+1)\frac{1}{a}.$$

More generally, it appears that $\mathrm{inv}(k, b)$ is the kth smallest value of $\mathrm{inv}(a, b)$ for $1 \leq k \leq 6$.

More specifically, the kth smallest value occurs with the remainder sequence

$$b,\ k,\ k-1,\ 1$$

The longer the remainder sequence $b, a, \ldots, 1$, the closer $\mathrm{inv}(a, b)$ is to $\frac{(b-1)(b-2)}{4}$.


Values of inv(a,b) vs. Length of remainder sequence for b=100000
5×10^9
4×10^9
3×10^9
2×10^9
1×10^9
5
10
15
20

Questions?